

Large N Duality, Mirror Symmetry, and a Q-deformed A-polynomial for Knots

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Abstract

We reconsider topological string realization of $SU(N)$ Chern-Simons theory on S^3 . At large N , for every knot K in S^3 , we obtain a polynomial $A_K(x, p; Q)$ in two variables x, p depending on the t'Hooft coupling parameter $Q = e^{Ng_s}$. Its vanishing locus is the quantum corrected moduli space of a special Lagrangian brane L_K , associated to K , probing the large N dual geometry, the resolved conifold. Using a generalized SYZ conjecture this leads to the statement that for every such Lagrangian brane L_K we get a distinct mirror of the resolved conifold given by $uv = A_K(x, p; Q)$. Perturbative corrections of the refined B-model for the open string sector on the mirror geometry capture BPS degeneracies and thus the knot homology invariants. Thus, in terms of its ability to distinguish knots, the classical function $A_K(x, p; Q)$ contains at least as much information as knot homologies. In the special case when $N = 2$, our observations lead to a physical explanation of the generalized (quantum) volume conjecture. Moreover, the specialization to $Q = 1$ of A_K contains the classical A-polynomial of the knot as a factor.

1 Introduction

The study of knot invariants has a long history in mathematics. With the introduction of Jones polynomials [1] and their physical interpretation in terms of Chern-Simons theories [2] and their embedding in topological strings [3] this has led to an interesting area of interaction between mathematics and physics in modern times. With the advent of string dualities new perspective was gained on this relation. Large N dualities for topological strings [4] lead to new predictions for HOMFLY polynomials. In particular the interpretation of topological strings as computing BPS degeneracies [5] has led to integrality predictions for colored HOMFLY polynomials. Furthermore, consideration of symmetries of the problem, leads to an interpretation of the Khovanov invariants [6] and knot homologies [7, 8], in terms of further decomposition of this BPS Hilbert space with respect to an additional $U(1)$ symmetry [9] (see also the recent work [10]). This refinement in turn relates to Nekrasov deformation of topological strings [11]. The deformation of the topological string also leads to a refinement of Chern-Simons theory recently discovered in [12].

In order to compute knot invariants using topological strings, one needs to get a good handle on the Lagrangian brane L_K associated to a knot, after the large N transition. Before the large N transition, the geometry of the Calabi-Yau is T^*S^3 and L_K is obtained from the knot K by the conormal bundle construction [5]. However, for general knots it has been difficult to use the large N duality to make concrete predictions about the knot invariants, as one needs a simple description of the L_K after the large N transition to the resolved conifold. For unknot this can be done explicitly, because it turns out that there is a mirror symmetry [13] which is compatible with the Lagrangian associated with the unknot and which maps the computation of the knot invariants to easy computations on the mirror. Until now it has been difficult to apply this idea to general knots and use topological strings as a practical method for computations of all knot invariants. One aim of this paper is to remedy this gap. We formulate a remarkable generalization of mirror symmetry in the spirit of SYZ [14], which allows us to compute, for the resolved conifold, a distinct mirror for each choice of knot K . We find that for every knot K in S^3 , the leading large N limit of the knot

invariants, leads to a polynomial

$$A_K(e^x, e^p, Q) = 0,$$

which captures the moduli of L_K corrected by disc instantons. The polynomial $A_K(e^x, e^p, Q)$ captures the mirror geometry in the spirit of SYZ. It determines the mirror Calabi-Yau Y_K , as a hypersurface

$$A_K(e^x, e^p, Q) = uv.$$

The polynomial $A_K(e^x, e^p, Q)$ can be viewed as an invariant of the knot. Mirror symmetry and large N-duality relate the computation of $SL(N)$ knot invariants to computation of open topological string amplitudes on the mirror Calabi-Yau Y_K , with a brane mirror to L_K . Nekrasov deformation of the topological string on Y_K is relevant for categorified knot invariants [9]. But, topological string amplitudes carry no more information about distinguishing Calabi-Yau than the classical geometry. This leads us to conclude that categorification carries no additional information for distinguishing knots beyond that captured in the polynomial $A_K(e^x, e^p, Q)$.

In a seemingly unrelated development, another polynomial for knots was introduced to physics in [15]. This is the A-polynomial of the knot, which characterizes flat $SL(2)$ holonomies on the knot complement (mathematical work on the A-polynomial originated in [16]). In [15, 17, 18] the generating polynomial for colored Jones was studied and conjectured to satisfy a difference equation (see also [19, 20]). These difference equations can be viewed as a quantum operator associated to the A-polynomial. This structure is reminiscent of difference equations satisfied by open topological string amplitudes [21] and in fact in some cases it was checked that the corresponding objects are canonically isomorphic, namely that a quantum deformation of the A-polynomial maps to a local geometry which in turn captures the topological string amplitudes [22, 23].

The secondary aim of this paper is to explain these observations in the context of the broader picture of the relation between knots and topological strings and to extend these observations to higher rank N . We find that the $A_K(e^x, e^p, Q) = 0$ geometry naturally computes the generating function relevant for the quantum volume conjecture. Moreover, the polynomial that captures the mirror associated to each knot turns out to be a Q -deformed version of the classical A-polynomial of the knot, where Q is related to the large N 't Hooft parameter. More precisely as $Q \rightarrow 1$ the mirror geometry

contains the A-polynomial of the knot as a factor. In this context, the fact that A_K is related to an operator annihilating the partition function of the knot, follows from general properties of open topological strings [21].

The plan for this paper is as follows: In section 2 we recall some basic aspects of mirror symmetry picture according to SYZ [14] and introduce an extension of it which is relevant for local non-compact Calabi-Yau. In section 3 we review the relation of Chern-Simons theory to topological strings on T^*S^3 and the Lagrangian L_K associated to a knot K . We also recall aspects of large N duality. In section 4, using large N duality, we sum up disk instanton corrections to the moduli space of the brane on the resolved conifold, and discover a family of mirrors, one for each knot K . In section 5 we review the A-polynomial for knots and relations to volume and AJ conjectures and derive these conjecture in the setup of topological string. Furthermore we interpret our Q-deformed A-polynomial as a generalization of the volume conjecture. In section 6 we give examples. In section 7, we discuss the relation between our polynomial and a similar one which has been proposed in the context of knot contact homology. In section 8 we make some concluding remarks and suggestions for future direction of research.

2 Mirror Symmetry in the Local Case

In this section we review certain features of mirror symmetry for non-compact Calabi-Yau manifolds. We will focus on the local toric Calabi-Yau three-folds, for which mirror symmetry is understood the best. This is also the case that naturally fits with knot theory, as we will see.

It is known [13, 24] that for the case of non-compact toric Calabi-Yau the mirror Calabi-Yau 3-fold takes the form

$$F(e^x, e^p) = uv \tag{2.1}$$

where $(e^x, e^p, u, v) \in \mathbf{C}^* \times \mathbf{C}^* \times \mathbf{C} \times \mathbf{C}$. In particular the main data of the Calabi-Yau is captured by the curve

$$F(e^x, e^p) = 0. \tag{2.2}$$

As discussed in [25] this curve can be viewed as the moduli space of a canonical special Lagrangian brane. These special Lagrangian submanifolds have the topology

of $S^1 \times \mathbf{R}^2$ and generalize the original constructions of Harvey and Lawson, and Joyce [26, 27]. By a theorem of McLean [28], a special Lagrangian with a $b_1 = 1$ has a one real dimensional moduli space coming from the deformations, and this is accompanied by the moduli of a flat bundle. If we parameterize the moduli space by $x = r + i\theta$, r comes from geometric deformations of the Lagrangian, and $\theta = \oint_{S^1} A$ from the choice of flat connection on the S^1 . The moduli space receives corrections from holomorphic disks ending on the brane. The disk amplitude $W(x)$ depends holomorphically on e^{-x} . The area of the disk is determined by the position r of the Lagrangian. Mirror symmetry relates the disk instanton corrections on the A-model side to classical periods of the mirror. As explained in [25], in the mirror, the disk amplitude $W(x)$ is the integral of the holomorphic threeform Ω on a chain whose boundary is the Lagrangian manifold, which for Calabi-Yau manifolds of the form (2.1) descends to a one form pdx on the curve (2.2).:

$$W(x) = \int^x p(x)dx.$$

The disk instanton corrected moduli space of these Lagrangian submanifolds is captured by a choice of a point on the mirror curve $F(e^x, e^p) = 0$. In other words we can parameterize the moduli space of the brane by the choice of x and find $p = p(x)$, such that $F(e^x, e^p) = 0$.

A simple example of this is the resolved conifold geometry given by

$$X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbf{P}^1.$$

Let $Q = e^{-t}$ where t denotes the complexified Kahler class of \mathbf{P}^1 . Then the mirror geometry is captured by the curve [13]

$$F(e^x, e^p) = 1 + e^x + e^p + Qe^{x+p} = 0.$$

This curve can, in turn, be viewed as the moduli space of the special Lagrangian branes on the resolved conifold. At one point along their moduli the S^1 of the Lagrangian is identified with the equator of \mathbf{P}^1 , while the \mathbf{R}^2 is a suitable subspace of the $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ vector bundle over it. Note that the classical moduli space of this Lagrangian brane has singularities, where the S^1 shrinks to a point. The singularities are smoothed out in the quantum theory by the disc instantons. In particular the geometry $F(e^x, e^p) = 0$

has no singular points, and this is a reflection of the fact that the classical geometry of the mirror manifold already captures the disk instanton sum on the A-model side.

In the SYZ formulation for mirror symmetry [14], in the compact case, one considers on the A-side a special Lagrangian with the topology of T^3 . This has a 3-complex dimensional moduli space. The mirror is identified as the quantum corrected moduli space, where the quantum corrections include the disc instantons. In the non-compact case it is natural to consider extensions of this where the special Lagrangian has a different topology. In particular from the discussion we just had, it is clear that a natural topology for the special Lagrangian to consider on the A-side is that of $\mathbf{R}^2 \times S^1$ and in this context the curve $F(e^x, e^p) = 0$ is naturally identified as the disk instanton corrected moduli space of the Lagrangian brane.¹

However the SYZ construction of mirror geometry admits a more general interpretation. Namely, we can consider *an arbitrary* special Lagrangian manifold L with the topology of $\mathbf{R}^2 \times S^1$. For each such choice, by summing up disk instanton corrections, we obtain a mirror geometry. The disc corrected moduli space is given by the curve

$$F_L(e^x, e^p) = 0,$$

which now depends on the Lagrangian L . Thus we end up with *infinitely* many mirrors for each toric manifold. In particular $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbf{P}^1$ can have infinitely many mirrors in the SYZ sense. The SYZ conjecture in this case becomes the statement that no matter which one of these choices of the B-model geometry we pick, we should nevertheless get identical topological string amplitudes. Moreover the open string amplitudes involving the Lagrangian L on the A-model side can be captured as the canonical brane on the B-model side in this mirror geometry – this brane is the brane that corresponds to a choice of a point on the curve, and the subspace $v = 0$ of the mirror Calabi-Yau

$$F_L(e^x, e^p) = uv.$$

In the next section, we will explain how, using the relation of topological string on $X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbf{P}^1$ to Chern-Simons theory on S^3 , and large N dualities, we

¹ The Lagrangian branes with the topology of $\mathbf{R}^1 \times T^2$ have also been considered in this context in [29, 30] for toric Calabi-Yau manifolds and relating the SYZ mirror picture with the mirror picture of [13].

can derive, for each knot K in the S^3 a distinct mirror Y_K mirror to X depending on the knot. The topological string on each such Y_K should be the same.

3 Topological String and Chern-Simons Theory

Consider topological A-model on

$$X^v = T^*S^3$$

with N Lagrangian D-branes on the S^3 . The partition function of the open A-model topological string is the $G = SU(N)$ Chern-Simons partition function on the S^3 [3],

$$Z^{top}(X, g_s) = Z_{CS}(S^3, k)$$

where the topological string coupling g_s coincides with the Chern-Simons coupling constant,

$$g_s = \frac{2\pi i}{k + N}.$$

A nice way to introduce knot invariants in this setup is to associate a particular Lagrangian L_K to the knot [5], so that

$$L_K \cap S^3 = K.$$

This is the conormal bundle construction, which basically means choosing the \mathbf{R}^2 planes of L_K to be in the cotangent fiber direction, and be orthogonal (relative to symplectic pairing) to the tangent to the knot. Adding a D-brane on L_K in the topological string corresponds to inserting

$$Z^{top}(X, L_K, x) = \langle \det(1 \otimes 1 - e^{-x} \otimes U)^{(-1)} \rangle_{S^3}$$

in Chern-Simons theory, where U is the holonomy along the knot, $U = P e^{i \oint_K \mathcal{A}}$. Since the Lagrangian L_K has topology $\mathbf{R}^2 \times S^1$, the theory on the brane has one modulus, which we will denote by x . The determinant captures the effect of integrating out the bifundamental scalar field of mass x , corresponding to the string stretching between the S^3 and L_K [5]. We can rewrite above as²

$$Z^{top}(X, L_K, x) = \sum_{n=0}^{\infty} \langle \text{Tr}_{R_n} U \rangle_{S^3} e^{-nx} \quad (3.1)$$

²This follows from the Cauchy formula in the theory of symmetric functions [31].

On the right hand side, the sum runs over R_n , the totally symmetric representations of $SU(N)$ with n boxes. Let

$$\langle Tr_{R_n} U \rangle_{S^3} = H_n(K),$$

denote the expectation value of the Wilson loop along the knot K , colored by representation R_n in the $SU(N)$ Chern-Simons theory.

For general $G = SU(N)$, the topological string partition function defines a wave function

$$Z^{top}(X, L_K, x) = \sum_n H_n(K) e^{-nx} = \Psi_K(x) \quad (3.2)$$

which is the exact partition function of the D-brane on L_K ,

$$\Psi_K(x) = \Psi_K(x, g_s, N)$$

depending on g_s and N . It is a wave function because L_K is a non-compact three manifold, with a T^2 boundary; x here corresponds to the holonomy around the S^1 in T^2 that remains finite in the interior of $L_K = \mathbf{R}^2 \times S^1$.

3.1 Large N duality

It is conjectured in [4] that, if we have N branes wrapped on S^3 in $X^v = T^*S^3$, at large N , the geometry undergoes a transition where S^3 shrinks and the N Lagrangian branes disappear, leaving behind a blown up \mathbf{P}^1 . In other words it is conjectured that the $SU(N)$ Chern-Simons theory is equivalent at large N to the topological string on

$$X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbf{P}^1.$$

where the t'Hooft coupling of Chern-Simons theory becomes the size of the \mathbf{P}^1 in X ,

$$Q = \exp(-t), \quad t = Ng_s.$$

The geometry of both X and X^v are cones over $S^2 \times S^3$. The geometric transition shrinks the S^3 at the apex of the cone in T^*S^3 and replaces it by the \mathbf{P}^1 , leaving the geometry far from the tip untouched.

Now consider the noncompact branes L_K we constructed in the previous section, one for each knot K in the S^3 . This Lagrangian gets pushed through the transition

to a Lagrangian L_K on $X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbf{P}^1$. We will abuse notation, and denote the Lagrangians on both X and X^v by the same letter, L_K . The topology of the Lagrangian is unchanged by the transition – L_K is still $\mathbf{R}^2 \times S^1$. To see this, note that we could have used the real modulus on the Lagrangian to push L_K away from the S^3 before the transition – this way, the Lagrangian can pass through the transition while remaining far from the apex of the cone where the geometric transition takes place. Thus, large N duality and the geometric transition gives us a way to construct a family of Lagrangians L_K of the topology of $\mathbf{R}^2 \times S^1$ in $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbf{P}^1$, one for each knot K in the S^3 .

The large N duality also gives a way to compute topological string amplitudes on X with and without branes. The partition function of Chern-Simons theory on the S^3 , as shown in [4], is the same as the closed topological string partition function of X . Similarly, the partition function of the theory on T^*S^3 with a D-branes on L_K gets related by duality to topological string on X with an A-brane on L_K . After the transition, one is computing open topological string amplitudes with contributions from holomorphic maps with boundaries on L_K . The N of Chern-Simons theory is captured by $Q = e^{-t} = q^N$. Q keeps track of the degrees of the holomorphic maps – their winding number around the \mathbf{P}^1 . In particular, we can use Chern-Simons theory to sum up the disk instanton corrections to the moduli space of the Lagrangian brane on L_K and, in this way derive the geometry of the mirror, as will be discussed in the next section.

4 Mirror Symmetry and Knot Invariants

We will now combine the various ideas discussed in the previous sections. We start with the T^*S^3 geometry and wrap N Lagrangian branes on the S^3 . This leads to $SU(N)$ Chern-Simons gauge theory on S^3 . We consider a knot K on S^3 and let L_K be the associated special Lagrangian brane which intersects the S^3 on K . As reviewed in section 3 at large N the T^*S^3 undergoes the geometric transition to the resolved conifold geometry with the modulus of the blown up \mathbf{P}^1 given by t where $Q = e^t$. Moreover the L_K will be represented by a special Lagrangian submanifold, which by abuse of notation we still denote by L_K . Note that L_K has the topology of $\mathbf{R}^2 \times S^1$ and so the discussion of the previous section applies. The theory has a mirror geometry

which is captured by a curve which we denote by

$$F_K(e^x, e^p; Q) = 0. \quad (4.1)$$

Moreover, as discussed in the previous sections, this classical geometry of the mirror captures all the disc corrections involving worldsheets ending on this special Lagrangian.

On the other hand, using geometric transition of the large N Chern-Simons theory, the exact open topological string partition function of the brane on L_K , in the resolved conifold, is equivalent to the computation of HOMFLY invariants (or knot homologies which arises in the Nekrasov deformation of topological strings and refined Chern-Simons theory [12, 32]). The exact wave function of the brane, computed by Chern-Simons theory and the topological string, is

$$\Psi_K(x, g_s, N) = \sum_n H_n(K) e^{-nx}. \quad (4.2)$$

The large N limit of the amplitude is equivalent to summing over all planar diagrams of the Chern-Simons theory. This is dominated by the disk instanton contribution that sums up all the planar graphs in Chern-Simons theory with one boundary on the D-brane. This can be written as

$$\Psi_K(x, g_s, Q) \sim \exp\left(\frac{1}{g_s} \int p_K(x, Q) dx\right). \quad (4.3)$$

Mirror symmetry and large N duality now imply that [5, 33]

$$p = p_K(x, Q) = \lim_{g_s \rightarrow 0} \sum_{n=0}^{\infty} g_s \langle \text{tr}_K U^n \rangle e^{-nx}$$

lives on the mirror curve (4.1). Conversely, from HOMFLY polynomial $H_{R_n}(q, N)$ colored by symmetric representations, and extracting the classical piece we can read off the curve of the mirror Calabi-Yau manifold. We will give examples of this in section 6.

4.1 Chern-Simons and The Quantum Mirror

Before we go on, note that instead of simply studying the classical equation of the mirror, Chern-Simons theory also gives us the exact quantum mirror geometry. In [21] it was found that the branes on the mirror Riemann surface are annihilated by

a quantum operator that quantizes the classical geometry. However, except when the underlying Riemann surface has genus zero, finding the exact brane wave functions is hard by direct methods, and correspondingly the quantum mirrors are not known in general. Large N duality and Chern-Simons theory gives us a way to find the exact brane wave function of the mirror. Thus, from the colored HOMFLY polynomial, without taking any limits, we can derive the quantum mirror Riemann surface.

The classical approximation to the wave function (4.3) determines the classical mirror geometry

$$F_K(e^x, e^p, Q) = 0.$$

As discussed in [21] this should be viewed as the classical $g_s \rightarrow 0$ limit of the operator equation

$$F_K(e^{\hat{x}}, e^{\hat{p}}, Q)\Psi_K(x) = 0 \quad (4.4)$$

where we view \hat{x} as a multiplication by x , and $\hat{p} = g_s \partial_x$, so that

$$[\hat{p}, \hat{x}] = g_s.$$

In this way Chern-Simons theory on the S^3 gives us a way to solve the topological string on the mirror of the brane and get the quantum Riemann surface associated to the knot. This is an example of quantum Riemann surfaces that emerge from studying D-branes in the topological string [21].

Before we go on, note that we can view $\Psi_K(x, Q)$ as a discrete Fourier transform of $H_{R_n}(K, q, Q)$. Correspondingly, an equivalent way to represent the quantum mirror curve is via its action on $H_{R_n}(K)$. This gives

$$\hat{F}_K(e^{\hat{x}}, e^{\hat{p}}, Q)H_{R_n}(K) = 0 \quad (4.5)$$

where now

$$e^{\hat{p}}H_{R_n}(K) = q^n H_{R_n}(K), \quad e^{\hat{x}}H_{R_n}(K) = H_{R_{n+1}}(K).$$

5 Relation to the A-polynomial and the Volume Conjecture

We have seen in the previous sections that mirror symmetry and large N duality associate a quantum Riemann surface of the form

$$\hat{F}_K(e^{\hat{x}}, e^{\hat{p}}; Q) = 0. \quad (5.1)$$

to a knot in S^3 . Quantum Riemann surfaces of this type have appeared in the literature in the context of the quantum volume and AJ conjectures. We will now explain what the connection is.

Consider Chern-Simons theory with gauge group $G = SU(2)$ and Chern-Simons level k on S^3 with a knot K inside it. The colored Jones polynomial $J_n(K)$ computes the expectation value of a Wilson loop along the knot, in the spin $n/2$ representation of G . It was noticed [15, 17] that the colored Jones polynomial $J_n(K)$ satisfies a difference equation,

$$\hat{A}_K(\hat{x}, \hat{p}) J_n(K) = 0, \quad (5.2)$$

where \hat{x} and \hat{p} are defined by

$$e^{\hat{x}} J_n = J_{n+1}, \quad e^{\hat{p}} J_n = q^n J_n,$$

where $q = e^{\frac{2\pi i}{k+2}} = e^{\hbar}$, where \hbar is the effective coupling constant of Chern-Simons theory.³ In particular,

$$[\hat{x}, \hat{p}] = \hbar. \quad (5.3)$$

Notice that $e^{\hat{p}} J_n = q^n J_n$ implies that p can be identified with ng_s . In fact this is most often how the wave function $\tilde{\psi}(p)$ of the $N = 2$ theory has been defined in the context of the volume conjecture. Namely by the analytic continuation of

$$\tilde{\Psi}(p) = J_{p/g_s}(K)$$

In the classical $\hbar \rightarrow 0$ limit A_K becomes a plane holomorphic curve,

$$A_K(x, p) = 0.$$

The curve that arises in this way is called the characteristic variety of the knot. Moreover, they are closely related to another Riemann surface from classical mathematics, the A-polynomial is defined by studying the moduli space of flat $SL(2, \mathbf{C})$ connections on the complement of the knot K in the S^3 . It was studied in [16]. Given a knot K in S^3 one considers its complement,

$$M_K = S^3/K$$

³In [17], one used the normalization in which the Jones polynomial of the unknot $J_n(\bigcirc) = 1$ for all n . We will not do that.

Cutting out the neighborhood of the knot from an S^3 , one obtains a three manifold with a T^2 boundary. The coordinates p and x parameterize the flat $SL(2, \mathbf{C})$ connections, around the cycles μ and λ that generate $\pi_1(T^2)$. Here μ is the meridian that links the knot, and the λ is the longitudinal cycle that runs parallel to the knot. The condition that these extend smoothly from the boundary T^2 to the interior of M_K imposes a relation between x and p that is the A-polynomial $A_K(x, p) = 0$. We now explain the relation to the Riemann surfaces $H_K(x, p, Q)$ that arise in the context of topological strings.

The colored Jones polynomial is a special case of the HOMFLY polynomial colored by symmetric representations, where we restrict to $N = 2$ and set $Q = q^N = q^2$,

$$J_n(K, q) = H_{R_n}(K, q^2, q).$$

This immediately implies that the quantum A-polynomial $A_K(\hat{x}, \hat{p})$ arises as a special case of the quantum mirror curve where we would replace $Q = q^2$. This identifies the corresponding quantum Riemann surfaces. The main difference between the descriptions of the wave functions is that in the context of the topological string the x coordinate, which is holonomy along longitudinal cycle of the knot is the natural variable, whereas in the context of the volume conjecture the p coordinate, which is the holonomy along the meridian is more natural. This in particular means that if we consider the Fourier transform of the wave function we have studied in the context of topological strings we would get the wave function relevant for the quantum volume or AJ conjecture:

$$\begin{aligned} \tilde{\Psi}_K(p) &= \int dx \exp\left(\frac{px}{g_s}\right) \Psi_K(x) \\ &= \int dx \exp\left(\frac{px}{g_s}\right) \sum_n \exp(-nx) J_n(K) = J_{p/g_s}(K) \end{aligned}$$

Moreover we expect that as we take q and Q to 1, we obtain the classical A-polynomial. Furthermore, the explanation of why the classical A-polynomial is related to Jones polynomial and how it relates to the volume of the hyperbolic knot complement now follows because all are related to $SL(2)$ Chern-Simons as was already noted in [15]. In particular in the classical limit fixing the x, p holonomies and solving the $SL(2)$ Chern-Simons equations, lead to flat solutions and thus they should lead to the constraint

captured by the A-polynomial. There is a subtlety here, as it turns out that the limit $q \rightarrow 1$ leads to a polynomial which is not exactly the A-polynomial but has the A-polynomial as a factor. We can offer an explanation of this, namely these could be interpreted as different branches of the brane geometry where the Lagrangian brane L_K has different reconnections with branes on S^3 as well as possibly moved off of S^3 . For example, for the unknot we get in this limit $(1 - e^x)(1 - e^p) = 0$. One solution $e^x = 1$, where x corresponds to longitudinal direction along K , is part of the classical A-polynomial and signifies the branch where the brane L_K has reconnected with one of the branes on S^3 and has trivialized the holonomy around the non-trivial circle in L_K . The other branch $e^p = 1$ could signify the geometry where the L_K is lifted off of S^3 , in which case e^p is trivial because it corresponds to the contractible circle in $T^2 \subset L_K$. It would be interesting to study this multi-branch structure for more general knots, and predict the additional factors that multiply the classical A-polynomial.

Considering the limit of $q \rightarrow 1$ keeping Q fixed, leads to a deformation of the classical A-polynomial to a Q-deformed version. More precisely, we have the following sequence of limits:

$$\hat{F}_K(e^{\hat{x}}, e^{\hat{p}}; Q) \xrightarrow{q \rightarrow 1} F_K(e^x, e^p, Q) \xrightarrow{Q \rightarrow 1} A_K(e^x, e^p)$$

and

$$\hat{F}_K(e^{\hat{x}}, e^{\hat{p}}; Q) \xrightarrow{Q \rightarrow q^2} \hat{A}_K(e^{\hat{x}}, e^{\hat{p}}) \xrightarrow{q \rightarrow 1} A_K(e^x, e^p)$$

More precisely in the last identity we expect that $A_K(e^x, e^p)$ is a factor of $F_K(e^x, e^p; 1)$. Thus, it simply follows from large N duality arguments and mirror symmetry that the mirror curve associated to the knot K , $F_K(e^x, e^p, Q)$ is a Q-deformation of a polynomial which has the classical A-polynomial of the knot $A_K(e^x, e^p)$ as a factor. This is the same as the classical mirror geometry associated to K . In other words, it simply follows from large N duality arguments and mirror symmetry that there exists a Q-deformation of a polynomial $A_K(e^x, e^p)$ which has the classical A-polynomial as a factor, which in turn is the same as the classical mirror geometry associated to K .

We can already make a few checks about this statement. In particular let us consider the limit $Q \rightarrow 1$. In this case we should obtain the mirror to the conifold at the singular point, where the size of the \mathbf{P}^1 is zero. If $A_K(e^x, e^p)$ is in fact the mirror geometry for the conifold, then it better be true that the periods we compute for A using

the mirror 1-form pdx are zero. More precisely the period should be zero, up to the ambiguity of adding an integer multiple of $2\pi i$, because $Q = 1$ only leads to $t = 2\pi i n$ for some n . Indeed this is one of the deep properties⁴ of the classical A-polynomial for knots [34]! This is already a first indication of the power of our conjecture, which automatically provides an alternative explanation of this fact. For general Q , the mirror curve $A_K(Q) = 0$ should have the same $\log(Q)$ -periods as the canonical mirror of the conifold. We will show this in the example of (m, n) torus knot curves, for arbitrary (m, n) . Unlike the A-polynomials, the $A_K(Q)$ polynomials of these curves are highly non-trivial, as we will see. Hence this is a strong test of our conjecture.

6 Examples

As we discussed above, the canonical of mirror of $X^v = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbf{P}^1$ is

$$uv = (1 - e^p) - e^x(1 - Qe^p).$$

In this section, we will explain that the canonical mirror is associated to the simplest knot in the S^3 , the unknot. Then, by changing the knot type, we will give examples of different mirrors that can arise. For each of these, we conjecture that topological strings on them are indistinguishable.

6.1 The Unknot

The canonical mirror Riemann surface,

$$F(e^x, e^p, Q) = (1 - e^p) - e^x(1 - Qe^p) = 0$$

is a quantum corrected moduli space of a simple Lagrangian brane $L_O = \mathbf{R}^2 \times S^1$ in $X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbf{P}^1$, where the S^1 can be roughly thought of as wrapping around the equator of the \mathbf{P}^1 , and where the \mathbf{R}^2 is the Lagrangian subspace of the $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ fiber to the \mathbf{P}^1 . From our previous discussion it is natural to ask which knot in the S^3 is this associated to. As explained in [5], the Lagrangian brane L_O in X^v corresponds to the unknot in the S^3 . The mirror geometry could have been derived from $G = SU(N)$ Chern-Simons theory along the lines we described above.

⁴ More precisely, they are known to have rational periods, which we interpret as defining the correct cover of the variables to lead to integral periods.

Recall that for the unknot, the HOMFLY polynomial colored by the n 'th symmetric representation R_n equals

$$H_n(\bigcirc) = S_{0R_n} = \frac{[N][N+1]\dots[N+n-1]}{[1][2]\dots[n]}$$

where S_{RQ} is the S matrix of $SU(N)_k$ Chern-Simons theory, and we need its element with $Q = 0$, $R = R_n$. We can rewrite this, in terms of $Q = q^N$ as follows:

$$H_n(\bigcirc) = Q^{-\frac{n}{2}} q^{\frac{n}{2}} \frac{(1-Q)(1-Qq)\dots(1-Qq^{n-1})}{(1-q)(1-q^2)\dots(1-q^n)} \quad (6.1)$$

where $[n] = q^{n/2} - q^{-n/2}$. From the recurrence relation satisfied by H_n

$$(1 - q^n)H_n - Q^{-\frac{1}{2}} q^{\frac{1}{2}} (1 - Qq^{n-1})H_{n-1} = 0.$$

we can read off the equation of the quantum mirror. Up to a redefinition of the holonomy variable x by a constant shift, we get

$$\hat{F}_{\bigcirc}(e^{\hat{x}}, e^{\hat{p}}, Q) = 1 - e^{\hat{p}} - e^{\hat{x}}(1 - Qe^{\hat{p}}) = 0$$

acting on H_n , or equivalently, on the wave function

$$\Psi_O(x) = \sum_{n=0} H_n e^{-nx}.$$

In particular, in the classical limit, $\hat{F}_{\bigcirc}(e^x, e^p, Q)$ gives the Riemann surface of the canonical mirror Calabi-Yau Y .

6.2 Torus knots

For any knot K we expect, via large N duality and mirror symmetry, to obtain a mirror Calabi-Yau to X . A large family of examples is provided by torus knots.

The colored $SU(N)_k$ polynomial for an (m, n) torus knot is given by Rosso-Jones formula [1]. It will be useful for us to recall a derivation of it, as understanding it will allow us to make some shortcuts later on. If we view the S^3 as a T^2 fibration over an interval, we can take an (m, n) torus knot to wind around the (m, n) cycle of the T^2 . This provides a way to compute the corresponding knot invariants: we think about the holonomy

$$\mathrm{Tr}_R U_{(m,n)} = \mathrm{Tr}_R e^{mx+np}$$

as the holonomy of an unknot knot winding m times around the $(1, 0)$ cycle, but with fractional framing $f = m/n$, corresponding to the shift of x to $x + m/np$.

$$\langle \text{Tr}_R e^{mx+np} \rangle = \langle \text{Tr}_R e^{n(x+m/np)} \rangle$$

Using the fact that an unknot in framing f , has expectation value $\langle \text{Tr}_Q e^{x+fp} \rangle = S_{0Q} T_Q^f$, where S and T are the S and the T matrices of $SU(N)_k$ Chern-Simons theory, and expanding $\text{Tr}_R U^n$ in terms of $\text{Tr}_Q U$

$$\text{Tr}_R U^n = \sum_Q C_{RQ}^{(n)} \text{Tr}_Q U.$$

we get the Rosso-Jones formula.

$$H_R^{(m,n)} = \langle \text{Tr}_R U_{(m,n)} \rangle = \sum_P C_{RP}^{(n)} T_P^{m/n} S_{P0}. \quad (6.2)$$

To find the D-brane moduli space, we can proceed in two equivalent ways. We can use the exact HOMFLY to find the quantum operator $\hat{F}_{(m,n)}(e^{\hat{x}}, e^{\hat{p}}, Q)$ annihilating the brane wave function

$$\hat{F}_{(m,n)}(e^{\hat{x}}, e^{\hat{p}}, Q) \Psi(x) = 0$$

and then take the classical g_s to zero limit to find the equation of the mirror. However, this is an overkill, if all we want is the classical geometry of the mirror. Moreover, should one want to find the exact recurrence, one would have to rewrite the colored HOMFLY in some way, to have a more compact formula for the n -th term of the sum. It would be very interesting to obtain such a formula, analogously to what was done for the figure 8 knot in [35] and recently in [36]⁵. Instead, one can simply use the leading large N limit of the HOMFLY amplitude, which was obtained in [38].⁶ For the (m, n) torus knot, one finds

$$\Psi(x) \sim \exp\left(\frac{1}{g_s} \int p(x) dx\right)$$

⁵As we prepared this paper for publication, [37] appeared, where just such a formula was found, and in fact extended to the refined setting. Using the formulas in [37], one could in fact obtain also the quantum mirror curves.

⁶In [38] the authors studied mirrors of torus knots. The curves obtained there are different than the ones we find. The difference, roughly, is that in their case, the mirror brane corresponds not to a single point on the Riemann surface as is the case for us, but to n points moving together where n is determined by the (m, n) type of the torus knot. The authors showed that one can reproduce the g_s corrections to some closed and open string amplitudes of the conifold with the branes corresponding to this knot. It would be nice to elucidate the physical interpretation of the curves obtained in that paper, as one is sure to exist.

where

$$p(x) = \sum_{k=0}^{\infty} V_k^{(m,n)} e^{-kx} \quad (6.3)$$

and [38, 39]

$$V_k^{(m,n)} = \frac{1}{(kn)!} \sum_{j=0}^{kn} \binom{kn}{j} (-1)^{kn+j} Q^{j+km-kn} \prod_{i=-j+1}^{kn-1-j} (km-i) \quad (6.4)$$

While these expressions do not have a manifest symmetry corresponding to exchanging (m, n) , once the sum in (6.4) is performed, the symmetry is restored. As we explained in section 4, the coefficients $V_k^{(m,n)}$ have a direct Chern-Simons interpretation, as a limit of the expectation value of the Wilson loop along the knot

$$V_k^{(m,n)} = \lim_{g_s \rightarrow 0} \langle \text{tr} U_{(m,n)}^k \rangle_{g_s}.$$

From (6.3) and (6.4) for specific m, n we can write down the mirror curves this gives rise to, by simply asking for $(x, p(x))$ to lie on the curve. We will explain later the method we used to obtain the curves, but for now it suffices to give examples of the results one gets.

6.2.1 The $(2, 3)$ knot

The simplest non-trivial torus knot is the $(2, 3)$ knot, the trefoil. For the trefoil, we find that $(x, p(x))$ lie on the following curve:

$$F^{(2,3)}(\alpha, \beta, Q) = (1 - Q\beta) + (\beta^3 - \beta^4 + 2\beta^5 - 2Q\beta^5 - Q\beta^6 + Q^2\beta^7) \alpha + (-\beta^9 + \beta^{10}) \alpha^2$$

For legibility of the formulas, in this and the following examples, we will denote

$$e^x = \alpha, \quad e^p = \beta.$$

To be precise, to reproduce the leading large N limit of the colored HOMFLY polynomial, we need to replace α by $\alpha Q^{5/2}$. While fractional powers of Q are natural in knot theory, they are not natural in the topological string, so we prefer to present the curve in this way. At $Q = 1$ limit, this becomes

$$F^{(2,3)}(\alpha, \beta, 1) = (-1 + \beta) (1 + \alpha\beta^3) (-1 + \alpha\beta^6)$$

The classical A polynomial of the trefoil knot,

$$A_{(2,3)}(\alpha, \beta) = (1 + \alpha) \left(-1 + \alpha\beta^3 \right)$$

corresponds to the last two factors in $F_{(2,3)}(\alpha, \beta, 1)$, after the change of framing that takes α to $\alpha\beta^{-3}$. In section 5 we suggested the interpretation of the remaining factor, $(-1 + \beta)$, as corresponding to the branch of the moduli space where the branes on L_K move off the S^3 . On this branch, the corresponding cycle becomes contractible in L_K , so the holonomy around it vanishes, setting β to 1.

6.2.2 The $(2, 5)$ knot

For the $(2, 5)$ knot, we get the following curve.

$$\begin{aligned} F^{(2,5)}(\alpha, \beta, Q) = & (1 - Q\beta) + (2\beta^5 - \beta^6 - Q\beta^6 + 3\beta^7 - 4Q\beta^7 + Q^2\beta^7 - 2Q\beta^8 + 2Q^2\beta^8 \\ & + 2Q^2\beta^9 - 2Q^3\beta^9 - Q^3\beta^{10} + Q^4\beta^{11})\alpha + (\beta^{10} - \beta^{11} + 2\beta^{12} \\ & - 2Q\beta^{12} - 2\beta^{13} + 2Q\beta^{13} + 3\beta^{14} - 4Q\beta^{14} + Q^2\beta^{14} - Q\beta^{15} - Q^2\beta^{15} \\ & + 2Q^2\beta^{16})\alpha^2 + (-\beta^{20} + \beta^{21})\alpha^3 \end{aligned}$$

This reproduces the HOMFLY, up to a shift of α by $Q^{7/2}$. In the $Q = 1$ limit, the curve degenerates to

$$F^{(2,5)}(\alpha, \beta, 1) = (-1 + \beta) (1 + \alpha\beta^5)^2 (-1 + \alpha\beta^{10})$$

The classical A-polynomial is a factor in $F^{(2,5)}(\alpha, \beta, 1)$, after one changes the framing by replacing $\alpha \rightarrow \alpha\beta^{-5}$.

6.2.3 The (2, 7) knot

For the (2, 7) knot, we get the following curve.

$$\begin{aligned}
F^{(2,7)}(\alpha, \beta, Q) = & (1 - Q\beta) + (3\beta^7 - \beta^8 - 2Q\beta^8 + 4\beta^9 - 6Q\beta^9 + 2Q^2\beta^9 - 3Q\beta^{10} \\
& + 4Q^2\beta^{10} - Q^3\beta^{10} + 3Q^2\beta^{11} - 4Q^3\beta^{11} + Q^4\beta^{11} - 2Q^3\beta^{12} + 2Q^4\beta^{12} \\
& + 2Q^4\beta^{13} - 2Q^5\beta^{13} - Q^5\beta^{14} + Q^6\beta^{15})\alpha + (3\beta^{14} - 2\beta^{15} - Q\beta^{15} + 6\beta^{16} \\
& - 8Q\beta^{16} + 2Q^2\beta^{16} - 3\beta^{17} + 2Q\beta^{17} + Q^2\beta^{17} + 6\beta^{18} - 12Q\beta^{18} + 10Q^2\beta^{18} \\
& - 4Q^3\beta^{18} - 3Q\beta^{19} + 2Q^2\beta^{19} + Q^3\beta^{19} + 6Q^2\beta^{20} - 8Q^3\beta^{20} + 2Q^4\beta^{20} \\
& - 2Q^3\beta^{21} - Q^4\beta^{21} + 3Q^4\beta^{22})\alpha^2 + (\beta^{21} - \beta^{22} + 2\beta^{23} - 2Q\beta^{23} - 2\beta^{24} \\
& + 2Q\beta^{24} + 3\beta^{25} - 4Q\beta^{25} + Q^2\beta^{25} - 3\beta^{26} + 4Q\beta^{26} - Q^2\beta^{26} + 4\beta^{27} - 6Q\beta^{27} \\
& + 2Q^2\beta^{27} - Q\beta^{28} - 2Q^2\beta^{28} + 3Q^2\beta^{29})\alpha^3 + (-\beta^{35} + \beta^{36})\alpha^4
\end{aligned}$$

In the $Q = 1$ limit, we get

$$F^{(2,7)}(\alpha, \beta, 1) = (-1 + \beta) (1 + \alpha\beta^7)^3 (-1 + \alpha\beta^{14})$$

The classical A-polynomial is a factor in $F^{(2,7)}(\alpha, \beta, 1)$, after one changes the framing by replacing $\alpha \rightarrow \alpha\beta^{-7}$.

6.3 Periods of the Torus Knot Curves

Our claim is that for every knot, in particular for every (m, n) torus knot, the curve gives rise to a mirror Calabi-Yau

$$uv = F^{(m,n)}(e^x, e^p, Q)$$

of the resolved conifold. From the SYZ perspective, all the mirrors are on the same footing, as there is no a-priori reason to favor one curve over the other. Note that the curves one gets are very high genus. The $F^{(2,3)}$ is nominally a genus three curve, while $F^{(2,5)}$ is a genus ten curve. Yet, we expect the topological strings on them to be equivalent. In particular, the classical periods of these curves should be equivalent. We will now prove that in fact for *all* torus knots, the latter statement holds. Moreover, the methods we will use to prove this will also be closely related to how we found the curves above, so we will have an opportunity to explain this.

To show that the periods of

$$\int p(x)dx$$

on a curve

$$F^{(m,n)}(e^x, e^p, Q) = 0$$

are the same as the periods of the conifold, we will find suitable variables in terms of which we can rewrite as

$$\int p(x)dx = \frac{1}{n} \sum_{k=0}^{n-1} \int p_k(x_k)dx_k$$

where each of the one forms $\int p_k(x_k)dx_k$ in fact has the same periods, and indeed the same periods as on the mirror of the conifold. This, together with the $1/n$ factor out front, will imply that $\int p(x)dx$ itself has the same periods, as the one form on the canonical mirror, thus proving our claim. To see how such variables may arise we need to back up, all the way to the beginning and the derivation of the Rosso-Jones formula.

In the presence of a Lagrangian brane corresponding to an (m, n) torus knot, the topological string computes the expectation value of

$$\Psi(e^x) = \langle \det^{-1}(1 \otimes 1 - U^n \otimes e^{-x}) \rangle$$

where $U = e^{x+m/np}$ corresponds to an unknot in the fractional framing. We can write this as

$$\Psi(e^x) = \langle \prod_{\ell=1}^n \det^{-1}(1 \otimes 1 - \omega^\ell U \otimes e^{-x/n}) \rangle$$

where ω is a primitive, n -th root of unity. As long as we are interested in the semi classical limit (or more precisely, as long as we are interested in single trace contributions to the amplitude where the multi-trace contributions are suppressed relative to this by powers of g_s), we can think of this as a wave function of n particles, evenly distributed

$$\Psi(e^x) \sim \prod_{\ell=1}^n \langle \det^{-1}(1 \otimes 1 - \omega^\ell U \otimes e^{-x/n}) \rangle = \prod_{\ell=1}^n \Psi_*(e^{-x/n} \omega^\ell)$$

Each of the n particles simply corresponds to an unknot in fractional framing $f = m/n$. If we denote their coordinates and momenta as (x_ℓ, p_ℓ) , $\ell = 1, \dots, n$ each of these lives

on the Riemann surface which is a simple mirror of the conifold, but with x shifted by $(m/n)p$, corresponding to the change of framing

$$e^{mp_\ell/n}(1 - e^{p_\ell}) - e^{-x_\ell}(1 - e^{p_\ell}Q) = 0.$$

The closed periods are independent of the change of framing [40] so in fact each of the one forms $\int p(x_\ell)dx_\ell$ has the same periods as the conifold. Now in order to relate this to $\Psi(x)$ we need to identify $e^{x_\ell} = e^{x/n}\omega^\ell$. This identification is not in $SL(2, \mathbf{Z})$ – it rescales pdx by $1/n$, so in fact each particle has the same periods as the canonical mirror of the conifold, albeit rescaled by $1/n$. But, this is exactly what we need to recover the periods of the mirror. In summary, while $(p(x), x)$, for a generic (m, n) torus knot live on a complicated curve, there is a way to introduce n sets of variables (p_ℓ, x_ℓ) in terms of which it becomes apparent that one is in fact studying the conifold.

This idea also helps in finding the explicit curve on which p and x live, $F^{(m,n)}(p, x)$. Namely, as long as we are interested in obtaining the classical curve only, we are allowed to replace $\Psi(e^x)$ by $\prod_{\ell=1}^n \Psi_\ell(e^{x_\ell})$. The latter has the form of a q -hypergeometric multi-sum. There are by now very powerful computer programs that can be used to find the corresponding q -recurrence relations. Taking the classical limit of this, gives us the curves we need. Explicitly, we have the brane amplitude of the framed unknot:

$$\Psi^{(f)}(x) = \sum_{R_k} T_{R_k}^f S_{R_k 0} e^{-kx} = \sum_{k=0} H_k(\bigcirc) q^{-fk^2/2} q^{fk(N-1)/2} e^{-kx}$$

where $H_k(\bigcirc)$ is the HOMFLY polynomial of the unknot (6.1). Taking $f = m/n$ gives $\Psi_*(x)$ we used, for a 1-particle wave function in our n -particle system.

6.4 Figure Eight Knot

The figure eight knot HOMFLY polynomial, colored by symmetric representation with n boxes is given by

$$\begin{aligned} H_n^{41} &= \sum_{i=0}^n \sum_{j=0}^i (-1)^i Q^{\frac{3+2i-3n}{2}} q^{\frac{-2n^2+i^2+4in+2j+7n-5i-5}{2}} \\ &\times \frac{[(q^{-i}, q)_j]^2 [(q^{1-n}, q)_i]^2 (Qq^{-1-i+j+n}, q)_{i-j} (Q, q)_{n-1}}{(q, q)_i (q, q)_j (q, q)_{n-1} (q^{n-i+j}, q)_{i-j}} \end{aligned}$$

This result was provided to us by K. Kawagoe [35], based on unpublished work. (see also the related work [41]). Recently, another expression for the colored HOMFLY in these

representations was obtained in [36]. The formulas should be equivalent, although they are not manifestly so. In the above, $(a, q)_n = \prod_{k=0}^{n-1} (1 - aq^k)$ is a version of q -factorial. The knot invariant is in the form of q -hypergeometric multi-sum. Recurrences of such sums can be found by a powerful program for finding recurrences of such sums, "HolonomicFunctions.m"⁷.

From this, we can find the mirror geometry corresponding to the figure eight knot.

$$\begin{aligned} F_{4_1}(\alpha, \beta, Q) &= (\beta^2 - Q\beta^3) + (-1 + 2\beta - 2Q^2\beta^4 + Q^2\beta^5)\alpha \\ &\quad + (1 - 2Q\beta + 2Q^2\beta^4 - Q^3\beta^5)\alpha^2 + (-Q^2\beta^2 + Q^2\beta^3)\alpha^3 \end{aligned}$$

This is obtained as the $q \rightarrow 1$ limit of the quantum \hat{F}_{4_1} polynomial. Viewed as a limit of the quantum curve, one finds additional trivial pieces, $(-1 + \beta^2)^2(\alpha - Q)^2$ terms. At $Q = 1$, our curve becomes

$$F_{4_1}(\alpha, \beta, 1) = (-1 + \alpha)(-1 + \beta)(\alpha - \alpha\beta - \beta^2 - 2\alpha\beta^2 - \alpha^2\beta^2 - \alpha\beta^3 + \alpha\beta^4)$$

This agrees with the A-polynomial of the figure eight knot, up to a factor $(-1 + \beta)$.

7 Relation with Knot Contact Homology

After the completion of this work, and during preparation of this manuscript, we became aware⁸ of the work by Ng [42] and collaborators [43], which seems to be related with the present work. For a readable review of the basic ideas see [44] and references therein. Their work uses symplectic field theory and knot contact homology. The relation of these with topological strings, Gromov-Witten theory and Chern-Simons theory was not observed previously. Below we propose an explanation of such a relation.

Both this paper and the ones by Ng and collaborators use the construction in [5] associating to each knot a Lagrangian brane. In the work [42] one starts with the Lagrangian brane in the T^*S^3 but deletes the zero section of the S^3 . In other, words, one is focusing on the geometry far from the apex of the cone where the geometry is

⁷The recurrence corresponding to this sum was computed by Christoph Koutschan, who also wrote the program. We are indebted to him for patiently explaining how to use his program. The program is publicly available from

<http://www.risc.jku.at/research/combinat/software/HolonomicFunctions/index.php>

⁸We thank Marc Culler, Craig Hodgson and Henry Segerman for pointing this out.

$S^3 \times S^2 \times \mathbf{R}$ which shares the basic feature of the geometry *after* the transition, where S^2 is no longer contractible. Furthermore one considers the L_K , i.e. the Lagrangian L_K associated with the knot, which has the topology of $T^2 \times \mathbf{R}$. It is natural to view \mathbf{R} as the ‘time’ direction. Here T^2 can be viewed as the geometry of the Lagrangian brane at infinity. The knot leaves its imprint on how T^2 is embedded in $S^3 \times S^2$. Then one considers the physical open string states ending on T^2 . These are in 1-1 correspondence with ‘Reeb chords’ a_i which are open string trajectories beginning and ending on T^2 and which extend along Reeb flows (which correspond to complex pairing of \mathbf{R} with a direction on $S^3 \times S^2$, and so the holomorphic maps can end on them). In the physical setup we would say that the a_i are classically annihilated by the BRST symmetry Q_B :

$$Q_B \cdot a_i = 0$$

However disk instantons could modify the operation of Q_B . This is similar to Witten’s formulation of Morse theory [45] where the critical points of the Morse function are in 1-1 correspondence with vacua, but instanton corrections, which correspond to gradient flows, modify the supersymmetry algebra. In the case at hand the role of the gradient flows are played by disc instantons. Consider one such disc D which at $t \rightarrow +\infty$ starts with the Reeb chord a_i and at $t \rightarrow -\infty$ approaches the Reeb chords a_{ij} as ordered multi-pronged strips with $j = 1, \dots, k_D$. Let the boundary of the disc map to $H_1(T^2)$ (upon choosing some fixed intervals capping the ends of the Reeb chords), which picks up the holonomy factor $\exp(n_i^D u + m_i^D v)$ from the Wilson lines on the Lagrangian brane. Let n_D denote the intersection of this disc with the 4-cycle dual to S^2 , i.e. $\mathbf{R} \times S^3$. Then Ng et.al. find a deformed Q_B operator (which we interpret as the quantum corrected Q_B) given by

$$Q_B \cdot a_i = \sum_{disc\ D} Q^{n_D} e^{n_i^D u + m_i^D v} a_{i_1} \dots a_{i_{k_D}}.$$

Thus the open string states correspond to non-trivial elements of the deformed Q_B cohomology. The existence of a 1-dimensional representations of this cohomology leads to a constraint $A(e^u, e^v, Q) = 0$. Such one dimensional representations can be interpreted as the disc corrected moduli space of one such Lagrangian brane after the large N transition. Using the large N duality this should be the same as the Q -deformed A-polynomial we defined, which was the quantum corrected moduli space of the single

brane L_K . We computed this *before* the transition using the color HOMFLY polynomial at leading order in large N . We have checked that for the examples $(2, 3)$, $(2, 5)$ torus knots and the figure-8 knot, for which the knot contact homology has been computed are in accord with computations done here⁹. This agreement is thus a consequence of the large N duality.

8 Concluding Remarks

The embedding of Chern-Simons theory in string theory and its large N duality lead to the statement that the knot invariants of the HOMFLY type as well as their Khovanov-type refinements can be deduced in principle once we know the Lagrangian brane associated with the knot on the large N dual geometry. In this paper, we have shown that, using a generalized SYZ conjecture adapted to the local Calabi-Yau geometries, to every knot we can associate a mirror Calabi-Yau Y_K . This mirror is captured by a classical polynomial $A_K(x, p; Q)$ which as $Q \rightarrow 1$ has the classical A-polynomial of the knot as a factor. Since HOMFLY and their refinement in the form of Khovanov invariants and knot homologies can be computed perturbatively starting with the classical B-model geometry of Y_K , it implies that the classical polynomial $A_K(x, p; Q)$ is at least as refined as the knot homologies.

Namely, having related a knot K to the Calabi-Yau Y_K , we relate quantum and homological invariants of a knot K to (open) topological string amplitudes on Y_K and their refinement. This implies that, to gain any new information about distinguishing knots by going to $SL(N)$ or Khovanov type invariants, we would need to get new information about distinguishing Calabi-Yau manifolds, beyond that captured by classical geometry, by considering Gromov-Witten or topological string amplitudes on Y_K instead. But, we do not: at least in principle, Gromov-Witten theory is a machine that uniquely assigns to a classical geometry, quantum amplitudes. We thus expect $A_K(x, y; Q)$ to completely capture all HOMFLY knot invariants, including its refinements. The one possible pitfall in this argument is the fact that the Calabi-Yau manifolds one obtains in this way are singular. Since knot invariants get related to open topological string amplitudes on Y_K and we already know that the disc amplitudes are not singular, and

⁹We thank Lenhard Ng for communicating his results.

moreover the closed string amplitudes are expected to be the same as those for the non-singular conifold, we do not expect the singularities to affect them. This point, given its importance, certainly deserves further study.

The fact that $A_K(x, p; Q)$ is a mirror of resolved conifold is striking. In particular A_K embeds as a 1-dimensional subspace of the mirror of a family of more complicated toric geometries. It would be interesting to understand how the amplitudes of closed topological strings of this more complicated geometry reduces to that of the conifold on this locus. In this paper we have checked some aspects of this relation for some knots, which would be nice to generalize to other knots.

The connection with knot contact homology seems interesting to explore further. The fact that knot contact homology and open Gromov-Witten theory are computing the same amplitudes was not known previously. The relation between knot contact homology and HOMFLY we uncovered is also new. In particular it would be nice to explicitly check that the disc corrected Q_B formulated in the context of A-model open topological strings agrees with what has been considered in the context of knot contact homology. Moreover the results in [42, 43] suggest that there is another algorithm to compute the Q-deformed A-polynomial using the braid representation of the knot. Since the HOMFLY polynomials can also in principle be computed using the braid representation, it would be nice to show that the large N limit of HOMFLY leads to the same result for the A_K^Q as that of the knot contact homology, thus verifying this aspect of large N duality prediction.

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